Stochastic Processes and their Applications 94 (2001) 29–49

stochastic processes and their applications

www.elsevier.com/locate/spa

Lyapunov exponents of Poisson shot-noise velocity fields

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Received 2 March 1998; received in revised form 31 July 2000; accepted 4 December 2000

Abstract

We consider the Lyapunov exponents of flows generated by a class of Markovian velocity fields. The existence of the exponents is obtained for flows on a compact set, but with the most general form of the velocity field. As a particular class, we study the homogeneous and incompressible flows. In this case, the exponents are nonrandom, free of the initial position of the particle path, and their sum is zero. We numerically compute the top Lyapunov exponent on $\mathbb{R}^2$ for a range of parameters to conjecture that it is strictly positive. © 2001 Elsevier Science B.V. All rights reserved.

Keywords: Stochastic flow; Lyapunov exponent; Poisson shot-noise; Stationary; Homogeneous; Turbulence

1. Introduction

In the last several years, significant research efforts have been directed toward the development of non-Gaussian velocity fields such as stable fields, functionals of Gaussian or stable fields, and fields described by nonlinear stochastic differential equations. Recently, a whole new class of random velocity fields has been introduced in Çinlar (1993) for modeling turbulent flows. Such velocity fields are functional versions of Poisson shot-noise and are close to those used in vortex methods. But they have superior qualities: they are stationary and ergodic and can be made incompressible and isotropic easily (Çinlar, 1994). They are meant to capture the medium scale structures observed in real oceanic flows, as opposed to the small scale structures modeled usually by Brownian or Gaussian type flows.

The velocity field is constructed as a superposition of deterministic velocity fields, which are randomized through their "types" and arrival times, types and times being governed by a Poisson random measure. Each arriving velocity field decays exponentially at a constant rate. The resulting cumulative velocity field is Markovian and stationary.

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PII: S0304-4149(01)00076-X

Our aim is to study the flows generated by Poisson shot-noise velocity fields through their Lyapunov exponents. The notion of Lyapunov exponents originates from the work of Lyapunov (1892) as a stability issue of the trajectories of a linear nonautonomous system

$$\dot{x} = A(t)x, \qquad x(0) = x_0 \in \mathbb{R}^d, \quad t \in \mathbb{R}_+, \tag{1.1}$$

where $A : \mathbb{R}_+ \to \mathbb{R}^{d\times d}$ is continuous and bounded, $\mathbb{R}^{d\times d}$ being the collection of real-valued $d\times d$ matrices, and the dot in $\dot{x}$ means differentiation with respect to t. The initial point x_0 is called *unstable* or *stable* depending on the exponential increase or decrease, respectively, of the length $\|x_t\|$ in time asymptotically. Precisely, the Lyapunov exponent of system (1.1) is defined as

$$\lambda(x_0) = \limsup_{t\to\infty} \frac{1}{t} \log \|x_t(x_0)\|.$$

It is well known that for a linear autonomous differential equation $\dot{x} = Ax$, where A is constant in time, all starting points x_0 are stable if and only if all the eigenvalues of A have negative real parts. On the other hand, the asymptotic properties of the solutions of (1.1) have little or no relationship with the eigenvalues of $A(t)$. This problem has led to the study of characteristic exponents which has applications in various dynamical systems (Arnold and Wihstutz, 1986).

The Lyapunov exponents of a system can be defined in terms of an associated matrix-valued function which relates to the stability of that system. In general, let M be a mapping from $\mathbb{R}_+$ to $\mathbb{R}^{d\times d}$. A real number λ is said to be a *Lyapunov exponent* for $\{M_t\colon t \geqslant 0\}$ if

$$\lambda = \lim_{t\to\infty} \frac{1}{t} \log \|M_t e\| \tag{1.2}$$

for some unit vector $e \in \mathbb{R}^d$. While e varies on S^{d-1}, the unit sphere in $\mathbb{R}^d$, the limit λ can take at most d different values. Let $\{e_1, e_2, \ldots, e_k\}$ be an orthonormal set in $\mathbb{R}^d$ and let K be the subspace spanned by them. The volume of the parallelepiped formed by the vectors $M_t e_1, \ldots, M_t e_k$, denoted by $V(M_t K)$, is called the *coefficient of expansion* in the direction of K and does not depend on the specific choice of $e_1, \ldots, e_k$. As a result, the notion of Lyapunov exponents can be generalized to k dimensions by

$$\lambda^{(k)} = \lim_{t\to\infty} \frac{1}{t} \log V(M_t K)$$

(Oseledec, 1968). The volume $V(M_t K)$ is given by the absolute value of the determinant of the transformation specified by the matrix M_t. In the linear equation (1.1), M_t is the fundamental matrix, that is, it satisfies $M_t x_0 = x(t)$, if $x(0) = x_0$. On the other hand, for a nonlinear system

$$\dot{x} = A(x,t), \qquad x(0) = x_0 \in \mathbb{R}^d, \quad t \in \mathbb{R}_+$$

with $A : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}^d$, the matrices of concern are the Jacobian matrices. That is, we define $M_t = [\partial x^i(t)/\partial x_0^j]$, $i, j = 1, \ldots, d$, and the space that each M_t acts upon is the tangent space.

In the analysis of Lyapunov exponents, ergodicity of both the Eulerian and the Lagrangian velocity fields play an important role. We use the term ergodicity in the context of stationary processes. Precisely, there exists a shift transformation ϑ associated with every stationary process X on $(\Omega, \mathscr{H}, \mathbb{P})$ (Rozanov, 1968). The shifts $\{\vartheta_s\colon s \in \mathbb{R}\}$ act on sets of $\sigma(X)$, the σ-algebra generated by X, and based on this, analogous shifts acting on the random variables in $\sigma(X)$ can be introduced. To denote the latter, we write $X_{s+t}(\omega) = X_t(\vartheta_s\omega)$, $\omega \in \Omega$, $s, t \in \mathbb{R}$, which notation emphasizes the meaning of ϑ_s, namely shifting the time origin to s. A set (event) A in $\sigma(X)$ is said to be *invariant* if $\vartheta_s A = A$ for all $s \in \mathbb{R}$. The collection of all invariant sets form the invariant σ-algebra $\mathscr{S}$. The shift $\{\vartheta_s\colon s \in \mathbb{R}\}$ is said to be *metrically transitive* if every event in $\mathscr{S}$ has probability 0 or 1. A stationary process is said to be *ergodic* (or *metrically transitive*) if its associated shift is metrically transitive.

The organization of the paper is as follows. The velocity field is reviewed in Section 2.1 and the flow is introduced in Section 2.2. In Section 3, we consider flows on a compact space and show that Lyapunov exponents exist in a general setting. In Section 4, we focus on homogeneous and incompressible flows. In Section 4.1, we show that the Lagrangian velocity field is also stationary and ergodic. As an important application of this result, the Lyapunov exponents are found to be nonrandom and free of the initial position in Section 4.2. Finally, in Section 4.3, we numerically compute the exponents of a homogeneous and incompressible flow on $\mathbb{R}^2$. The results support the conjecture that the exponents are distinct and hence the top exponent is strictly positive as should be in a turbulent flow.

2. Flows and the velocity field

In this section, we introduce the flows generated by Poisson shot-noise velocity fields and prove their basic properties. We first review some facts about the velocity field as given in Çinlar (1993, 1994).

2.1. The velocity field

Let $(\Omega, \mathscr{H}, \mathbb{P})$ be a probability space. Let Q be a topological space and let N be a Poisson random measure on the Borel subsets of $\mathbb{R} \times Q$ whose mean measure has the form

$$\nu(\mathrm{d}t, \mathrm{d}q) = \mathrm{d}t\, \kappa(\mathrm{d}q), \quad t \in \mathbb{R}, \ q \in Q,$$

where κ is a Borel measure on Q. The random velocity field $u(x,t)$ as a function of position $x \in \mathbb{R}^d$ and time $t \in \mathbb{R}_+$ is defined as

$$u(x,t) = \mathrm{e}^{-ct} u_0(x) + \int_{[0,t]\times Q} N(\mathrm{d}s, \mathrm{d}q) \mathrm{e}^{-c(t-s)} v_q(x), \tag{2.1}$$

where c is a strictly positive parameter, u_0 is the initial velocity field, and the v_q are continuous vector fields on $\mathbb{R}^d$ with $\int_Q \kappa(\mathrm{d}q)\|v_q(x)\| < \infty$ for every $x \in \mathbb{R}^d$. We will refer to v_q as a vortex of type q; this usage of the term “vortex” should not be confused with technical terms like vorticity. Each atom (t_i, q_i), $i = 1, 2, \ldots$, of N specifies the type q_i and the arrival time t_i of a vortex. As evident from Eq. (2.1), the velocity field is the Poisson driven version of Ornstein–Uhlenbeck type fields.

The distribution of $u_t \equiv \{u(x,t): x \in \mathbb{R}^d\}$ is characterized by its Fourier transform

$$\mathbb{E} \exp \mathrm{i} \int_{\mathbb{R}^d} \rho(\mathrm{d}x) \cdot u(x,t)$$

defined for $\mathbb{R}^d$-valued measures ρ on $\mathbb{R}^d$. When u_0 is independent of the Poisson random measure N, the distribution of the velocity field u_t converges to a unique stationary distribution with Fourier transform

$$\exp \int_0^\infty \mathrm{d}s \int_Q \kappa(\mathrm{d}q) \left[\exp\left\{ \mathrm{i}\mathrm{e}^{-cs} \int_{\mathbb{R}^d} \rho(\mathrm{d}x) \cdot v_q(x) \right\} - 1 \right] \tag{2.2}$$

(Çinlar, 1993, Proposition 3.31). Hence, the following form of u defines a stationary velocity field:

$$u(x,t) = \int_{(-\infty,t]\times Q} N(\mathrm{d}s, \mathrm{d}q)\mathrm{e}^{-c(t-s)} v_q(x), \quad x \in \mathbb{R}^d, \ t \in \mathbb{R}. \tag{2.3}$$

The means $\mathbb{E}\, u(x,t)$ and the covariances $\mathrm{Cov}(u(x,t), u(y,s))$, $x, y \in \mathbb{R}^d$, $s, t \in \mathbb{R}$ are also computed in Çinlar (1993). It follows from (2.1) that $\{u(x,t): t \in \mathbb{R}_+\}$ is a Markov process for each $x \in \mathbb{R}^d$, as the increments of the Poisson random measure N are independent.

Since N has stationary increments, it is possible to take $(\Omega, \mathscr{H}, \mathbb{P})$ such that there are shift operators ϑ_t with the following properties:

(i) $\vartheta_s \circ \vartheta_t = \vartheta_{s+t}$,
(ii) $(\omega, t) \to \vartheta_t \omega$ is measurable with respect to $\mathscr{H} \otimes \mathscr{B}_{\mathbb{R}}$ and $\mathscr{H}$,
(iii) $N(\vartheta_t\omega, \ A \times B) = N(\omega, (A+t) \times B)$.

Let $\mathscr{F}_d$ be the σ-algebra generated by the Poisson random measure N, that is

$$\mathscr{F}_d = \sigma\{N(A \times B): A \in \mathscr{B}_{\mathbb{R}}, \ B \in \mathscr{B}_Q\}.$$

Proposition 2.1. *The velocity field* $\{u(x,t): x \in \mathbb{R}^d, \ t \in \mathbb{R}\}$ *is ergodic.*

Proof. From Doob (1953, Theorem XI.1.1), it follows that Poisson random measure N is metrically transitive (or ergodic) with respect to $\mathscr{F}_d$, as it has stationary independent increments. In other words, the shift $\{\vartheta_t: t \in \mathbb{R}\}$ is metrically transitive. Now, we have

$$\begin{aligned}
u(\vartheta_t\omega, \cdot, 0) &= \int_{-\infty}^0 \int_Q N(\vartheta_t\omega, \mathrm{d}s, \mathrm{d}q)\mathrm{e}^{cs} v_q(\cdot) \\
&= \int_{-\infty}^0 \int_Q N(\omega, t + \mathrm{d}s, \mathrm{d}q)\mathrm{e}^{cs} v_q(\cdot) \\
&= \int_{-\infty}^t \int_Q N(\omega, \mathrm{d}\tilde{s}, \mathrm{d}q)\mathrm{e}^{-c(t-\tilde{s})} v_q(\cdot) \\
&= u(\omega, \cdot, t),
\end{aligned}$$

where we first used the stationarity of the increments of N and then made a change of variable $\tilde{s} = t + s$. That is, u is stationary with the same shift ϑ_t which is metrically transitive. Note that u is also in $\mathscr{F}_d$. Hence, u is ergodic. □

The vortices v_q and the form of the mean measure v of N determine the spatial properties of the velocity field (Çinlar, 1994). The velocity field u is called *homogeneous* in space if, for each $t \in \mathbb{R}$, the probability law of the collection $\{u(z+x,t) \colon x \in \mathbb{R}^d\}$ is the same for all $z \in \mathbb{R}^d$. The following theorem from Çinlar (1994) characterizes homogeneity.

Theorem 2.2. *Let $Q = \mathbb{R}^d \times \mathbb{R} \times (0,\infty)$ and let v_q be obtained from a deterministic velocity field v by*

$$v_q(x) = av\left(\frac{x-z}{b}\right) \quad \text{if } q = (z,a,b) \tag{2.4}$$

for $x \in \mathbb{R}^d$. Suppose that the measure κ on Q has the form

$$\kappa(\mathrm{d}q) = \mathrm{d}z\,\gamma(\mathrm{d}a,\mathrm{d}b) \quad \text{if } q = (z,a,b),$$

where γ is a finite measure on $\mathbb{R} \times (0,\infty)$. Then, $\{u(x,t) \colon x \in \mathbb{R}^d\}$ is homogeneous for each time t in $\mathbb{R}$.

A word on definition (2.4) is in order. The vortices v_q have one deterministic shape v up to random translations z, amplitudes a, and scales b, all governed by the Poisson random measure N. We call v the *basic vortex*.

2.2. The flow

The *flow* generated by the velocity field u is defined as the family of solutions of the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}X_t = u(X_t,t), \quad X_0 = x, \ \ t \in \mathbb{R}_+ \tag{2.5}$$

while x varies in $\mathbb{R}^d$. For fixed x, we interpret $\{X_t \colon t \geqslant 0\}$ as the path of a particle which started at x at time 0. Rigorously speaking, the flow generated by u is the family of transformations $\varphi_{s,t}$, $0 \leqslant s \leqslant t < \infty$, where $\varphi_{s,t}(x)$ is the solution of (2.5) for $t \geqslant s$ and $X_s = x$. We shall also write $\varphi(t)x$ for $\varphi_{0,t}x$. Since the flow arises as a solution of an ordinary differential equation, we shall use available results to show its existence and uniqueness. It will be seen that the discontinuity of u in the time variable presents no difficulty. The particle paths are continuous, as a result of the integration of (2.5). The following theorem summarizes the existence, uniqueness, and the diffeomorphic property of the solutions of (2.5). We state the sufficient conditions simply in terms of v_q, $q \in Q$, under definition (2.3). They can be modified in an obvious way to include u_0, when definition (2.1) is considered.

Theorem 2.3. *Suppose that each v_q is Lipschitz continuous, that is, there exists $M_q > 0$ such that for every $x, y \in \mathbb{R}^d$*

$$|v_q(x) - v_q(y)| \leqslant M_q|x-y|$$

and assume that $\int_Q \kappa(\mathrm{d}q)M_q < \infty$. *Then, there exists a unique continuous solution* X *of* (2.5) *on* $\mathbb{R}_+$ *for every* $x \in \mathbb{R}^d$ *almost surely. A bound for* $|X_t - x|$ *is*

$$|X_t - x| \leqslant \int_0^t \mathrm{d}s|u(x,s)| \exp \int_0^t \mathrm{d}s \int_{-\infty}^s \int_Q N(\mathrm{d}r,\mathrm{d}q)\mathrm{e}^{-c(s-r)}M_q.$$

If moreover $v_q \in C^1$ *for all* $q \in Q$, *then the transformations* $\{\varphi(t)\colon t \in \mathbb{R}_+\}$ *are diffeomorphisms of* $\mathbb{R}^d$ *almost surely.*

Proof. We outline the steps, the details can be found in Çağlar (1997). In every finite region $R \subset \mathbb{R}^d$, u_t is bounded by a right continuous Lebesgue integrable function. Moreover, it is Liptschitz continuous with constant $l(t)$ where l is also Lebesgue integrable on finite intervals. It follows from Filippov (1988) that the flow exists on R almost surely, is unique, and can be indefinitely continued over $\mathbb{R}^d$. The homeomorphic property follows from uniqueness and continuity of the solution with respect to the initial position x. Then proof is given in Theorem I.4.1 of Coddington and Levinson (1955). When $v_q \in C^1$ for all $q \in Q$, each $\varphi(t)$ becomes a diffeomorphism. □

We have seen that the velocity field is Markov and has a unique stationary distribution. On the other hand, the future evolution of a particle path depends on the initial velocity field in addition to its starting point. As a result, the one point motion $\{X_t\colon t \geqslant 0,\ X_0 = x\}$ is not a Markov process by itself; instead, we consider the joint process formed by the velocity field and the particle path.

We consider the process $Z = \{(u(\cdot,t),X_t)\colon t \in \mathbb{R}_+\}$. We assume that u_0 and v_q, $q \in Q$, and hence the velocity field $u_t \equiv \{u(x,t)\colon x \in \mathbb{R}^d\}$ vanish at the boundary and outside of a compact subset D of $\mathbb{R}^d$ for each $t \geqslant 0$. We also assume that u_0 and each v_q have continuous first partial derivatives, which is needed in the forthcoming sections. The state space of Z then becomes $E = C^1(D \to \mathbb{R}^d) \times D$ where $C^1(D \to \mathbb{R}^d)$ denotes the space of continuous functions from D to $\mathbb{R}^d$ with continuous first partial derivatives. The process Z is a time-homogeneous Markov process with respect to the filtration $\mathscr{F}_t = \sigma(X_0,\ u(\cdot,s)\colon 0 \leqslant s \leqslant t)$. Let P_t denote the transition semigroup of Z. We say that a sequence $(y_n) \subset C^1(D \to \mathbb{R}^d)$ converges to $y \in C^1(D \to \mathbb{R}^d)$ if and only if $\|y_n - y\|_\alpha \equiv \sup_{x \in D}|D^\alpha y_n(x) - D^\alpha y(x)|$ goes to 0 as $n \to \infty$ for all $|\alpha| = 1$ where

$$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial_{x_1}^{\alpha_1} \ldots \partial_{x_d}^{\alpha_d}}, \quad |\alpha| = \alpha_1 + \cdots + \alpha_d,\ \ \alpha_i \in \mathbb{N}.$$

The transition semigroup P_t satisfies a Feller type property given in the following proposition.

Proposition 2.4. *If* $f\colon E \to \mathbb{R}$ *is continuous and bounded, then so is* $P_t f$.

Proof. Let $y \in C^1(D \to \mathbb{R}^d)$ be an initial velocity field for u and (y_n) be a sequence converging to y. Let $(u_t^{(n)})$ denote the corresponding sequence at time t. Since

$$\sup_{x \in D} |D^\alpha u_t^{(n)}(x) - D^\alpha u_t(x)| \leqslant \mathrm{e}^{-ct} \sup_{x \in D} |D^\alpha y_n(x) - D^\alpha y(x)|,$$

we have $(u_t^{(n)}) \to u_t$. Let (x_n) be a sequence converging to x and let $X_t^{(n)}$ denote the solution of (2.5) with $u_0 = y_n$ and $X_0 = x_n$. Since u depends continuously on

y and x, it follows from Filippov (1988, Chapter 1, Theorem 6) that $\{X_t^{(n)}\} \to \{X_t\}$ almost surely. Hence, if $(y_n,x_n) \to (y,x)$, then $(u_t^{(n)},X_t^{(n)}) \to (u_t,X_t)$ almost surely. This implies that $\mathbb{E}f(u_t^{(n)},X_t^{(n)}) \to \mathbb{E}f(u_t,X_t)$ for all continuous bounded functions f by bounded convergence theorem. That is, $P_tf(y_n,x_n)$ converges to $P_tf(y,x)$. □

This following result is the basis of our results on the existence of Lyapunov exponents for the flow in the next section.

Proposition 2.5. *Suppose the velocity fields v_q, $q \in Q$ vanish at the boundary and outside of a compact subset D of $\mathbb{R}^d$. Then, there exists a stationary distribution for the Markov process $Z = \{(u_t,X_t)\colon t \in \mathbb{R}_+\}$.*

Proof. Take u as defined by (2.3). Then, all u_t, $t \geqslant 0$, have the stationary distribution characterized by (2.2) say ζ. Tightness of $\{u_t\colon t \geqslant 0\}$ follows trivially as $C^1(D \to \mathbb{R}^d)$ is separable and complete (Ethier and Kurtz, 1986, Lemma 3.2.1). That is, for each $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset C^1(D \to \mathbb{R}^d)$ such that we have $\zeta(K) \geqslant 1-\varepsilon$. We define a sequence of measures v_n on $E = C^1(D \to \mathbb{R}^d) \times D$ by

$$v_n(A \times B) = \frac{1}{t_n}\int_0^{t_n} \mathbb{P}\{u(\cdot\,,s) \in A,\ X_s \in B\}\,\mathrm{d}s, \quad A \in \mathscr{B}_{C^1(D \to \mathbb{R}^d)},\ B \in \mathscr{B}_{\mathbb{R}^d}$$

for some sequence (t_n) such that $t_n \to \infty$ as $n \to \infty$. For K_ε specified above $v_n(K_\varepsilon \times D) = \zeta(K_\varepsilon) \geqslant 1-\varepsilon$. Then, $\inf_n v_n(K_\varepsilon \times D) \geqslant 1-\varepsilon$, for each $\varepsilon > 0$, that is, (v_n) is tight. This implies that (v_n) is relatively compact by Prohorov's Theorem, as E is complete and separable. Therefore, (v_n) has a subsequence that converges. Let v denote the limit, and let us relabel the subsequence to be (v_n). Then v is a stationary distribution for Z, because for each $f \in C_b(E)$ we have

$$\begin{aligned}
\int_E v(\mathrm{d}z)P_tf(z) &= \lim_{n\to\infty}\int_E v_n(\mathrm{d}z)P_tf(z) = \lim_{n\to\infty}\frac{1}{t_n}\int_0^{t_n}\mathbb{E}[P_tf(Z_s)]\,\mathrm{d}s \\
&= \lim_{n\to\infty}\frac{1}{t_n}\int_t^{t+t_n}\mathbb{E}f(Z_s)\,\mathrm{d}s \\
&= \lim_{n\to\infty}\frac{1}{t_n}\left[\int_0^{t_n}\mathbb{E}f(Z_s)\,\mathrm{d}s - \int_0^t\mathbb{E}f(Z_s)\,\mathrm{d}s + \int_{t_n}^{t+t_n}\mathbb{E}f(Z_s)\,\mathrm{d}s\right] \\
&= \lim_{n\to\infty}\frac{1}{t_n}\int_0^{t_n}\mathbb{E}f(Z_s)\,\mathrm{d}s = \lim_{n\to\infty}\int_E v_n(\mathrm{d}z)f(z) = \int_E v(\mathrm{d}z)f(z),
\end{aligned}$$

where the first and last equalities follow from the interchange of the integral and the limit by the continuity of P_tf by Proposition 2.4 and the bounded convergence theorem. □

3. Lyapunov exponents on a compact space

In this section, we consider flows on a compact space. In order to apply Oseledec's theorem, we begin by identifying the underlying dynamical system induced by the flow. This system has similarities to that studied in Crauel (1986).

3.1. The dynamical system

Let $\{\vartheta_t\colon t \in \mathbb{R}_+\}$ be a family of measure preserving and ergodic transformations on $(\Omega, \mathscr{H}, \mathbb{P})$ such that $(\omega, t) \to \vartheta_t \omega$ is measurable. Then $(\Omega, \mathscr{H}, \mathbb{P}, (\vartheta_t)_{t\in T})$ is a dynamical system, and we define a random dynamical system as follows (Arnold and Crauel, 1991). A *random dynamical system* over $(\vartheta_t)_{t\in T}$ on $(\Omega, \mathscr{H}, \mathbb{P})$ is a family $\{\varphi(\omega, t)\colon \omega \in \Omega,\ t \in T\}$ of transformations on a measurable space $(E, \mathscr{E})$ such that, for each $t \in T$, the mapping $(\omega, x) \to \varphi(\omega, t)x$ is measurable and, for almost every ω,

$$\text{(i)}\ \ \varphi(\omega, 0) = \text{identity},$$

$$\text{(ii)}\ \ \varphi(\omega, t+s) = \varphi(\vartheta_s \omega, t) \circ \varphi(\omega, s) \tag{3.1}$$

for all $s, t \in T$. Condition (3.1)(ii) is called the cocycle property and hence φ is called a *cocycle* over ϑ.

A random dynamical system induces a skew product flow $\Theta_t\colon \Omega \times E \to \Omega \times E,\ t \in T$, where

$$\Theta_t(\omega, x) = (\varphi(\omega, t)x, \vartheta_t \omega). \tag{3.2}$$

The flow property $\Theta_{t+s} = \Theta_t \circ \Theta_s$ follows from that of ϑ_t and property (3.1)(ii) of φ. A probability measure μ on $(\Omega \times E, \mathscr{H} \otimes \mathscr{E})$ is said to be an *invariant measure* for φ if μ is invariant under the shifts Θ_t, $t \in T$, and if it has marginal $\mathbb{P}$ on Ω. Let $\varphi(\omega, t)x$ denote the solution of the following flow equation X:

$$\frac{\mathrm{d}}{\mathrm{d}t} X_t = u(\omega, X_t, t), \quad X_0 = x, \ \ t \in \mathbb{R} \tag{3.3}$$

on the stationary velocity field u. We will refer to the family of transformations $\{\varphi(\omega, t)\colon \omega \in \Omega,\ t \in \mathbb{R}\}$ on $\mathbb{R}^d$ as the flow, which choice is justified by the stationarity of the velocity field. Since φ is continuous in t and measurable in (ω, x), the mapping $(\omega, x, t) \to \varphi(\omega, t)x$ is also measurable. Let $\{\vartheta_t\colon t \geqslant 0\}$ be the shift transformation associated with u as in Section 2.1. By construction, ϑ_t preserves $\mathbb{P}$, and $(\omega, t) \to \vartheta_t \omega$ is measurable by the right continuity of u. The stationarity of the velocity field u with respect to the shift ϑ, and the almost sure uniqueness of the solution φ leads to the cocycle property (3.1)(ii). As a result, $\{\varphi(\omega, t)\colon \omega \in \Omega,\ t \in \mathbb{R}\}$ forms a random dynamical system over $(\vartheta_t)_{t\in\mathbb{R}}$ on $(\Omega, \mathscr{H}, \mathbb{P})$.

3.2. Multiplicative ergodic theorem

The dynamical system that we are concerned with has two parts: the shifts on the probability space and the flow on the real domain. This is an indication that the ergodicity of the velocity field by itself will not be sufficient for our purposes. Indeed, Lyapunov exponents are defined in terms of a *cocycle* with respect to the specific dynamical system. Let $\nabla_x \varphi(\omega, t)$ denote the Jacobian matrix of φ, that is

$$[\nabla_x \varphi(\omega, t)]_{ij} = \frac{\partial \varphi(\omega, t)^i x}{\partial x^j}.$$

Applying the chain rule to (3.1)(ii), we get

$$\nabla_x \varphi(\omega, t+s) = \nabla_{\varphi(\omega,s)x} \varphi(\vartheta_s \omega, t) \circ \nabla_x \varphi(\omega, s). \tag{3.4}$$

Consequently, $\nabla\varphi$ is a cocycle over the skew product flow $(\omega,x)\to\Theta_t(\omega,x)$ of (3.2). The Lyapunov exponents are defined by

$$\lambda(\omega,x)=\lim_{t\to\infty}\frac{1}{t}\log\|\nabla_x\varphi(\omega,t)e\| \tag{3.5}$$

for all possible choices of the unit vector e as given in Eq. (1.2). This definition arises essentially by the linearization of the nonlinear equation (3.3), since by the chain rule we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\nabla_x\varphi(t)=\nabla u(\varphi(t)x,t)\nabla_x\varphi(t). \tag{3.6}$$

In this section, let $\lambda_1(\omega,x)>\cdots>\lambda_r(\omega,x)$, $r(\omega,x)\leqslant d$, denote the distinct Lyapunov exponents, and $k_i(\omega,x)$ denote the multiplicity of $\lambda_i(\omega,x)$, $i=1,\dots,r(\omega,x)$. The multiplicative ergodic theorem (MET) of Oseledec (1968) is the basic tool that provides the conditions under which the Lyapunov exponents exist. First, we need measurability of Θ in (ω,x,t). In the previous subsection, we have discussed the measurability of the mappings $(\omega,t)\to\vartheta_t\omega$ and $(\omega,x,t)\to\varphi(\omega,t)x$ and $(\omega,t)\to\vartheta_t\omega$. Consequently, $(\omega,x,t)\to\Theta_t(\omega,x)$ is measurable. The next step is the identification of an invariant measure for the transformation Θ. In Section 2.2, we showed that when φ are transformations on a compact set $D\subset\mathbb{R}^d$, the transition semigroup P_t of the Markov process $Z=(u_t,X_t)$ has a stationary distribution v. We consider the random dynamical system φ on the smooth compact $D\subset\mathbb{R}^d$. We characterize an invariant measure for φ, in fact for the flow Θ_t on $\Omega\times D$. Let α denote the disintegration of v on D with respect to ζ, that is, $v(\mathrm{d}y,\mathrm{d}x)=\alpha(y,\mathrm{d}x)\zeta(\mathrm{d}y)$ (since D is complete and separable, α exists). Then, the measure μ defined by

$$\mu(\mathrm{d}\omega,\mathrm{d}x)=\alpha(u_0(\omega),\mathrm{d}x)\mathbb{P}(\mathrm{d}\omega)$$

on $\Omega\times D$ is invariant under Θ_t, $t\geqslant 0$.

The following is the multiplicative ergodic theorem for our system. For simplicity, let us denote $V(M_tK)$ of Section 1 by $J_t(K)$ where $M_t=\nabla_x\varphi(t)$. Let ∇u denote the matrix $[\partial u^i/\partial x^j]$ ($i=$ row, $j=$ column), and T_xD denote the tangent space of D at x. The *norm of a matrix* A is defined as $\|A\|=\max_{\{e\,:\,\|e\|=1\}}\|Ae\|$.

Theorem 3.1. *If*

$$\|\nabla u(\cdot,\cdot,0)\|\in L^1(\Omega\times D,\mu), \tag{3.7}$$

then there exists $\Gamma\in\mathscr{H}\otimes\mathscr{B}_D$ with $\mu(\Gamma)=1$ such that for every $(\omega,x)\in\Gamma$ the following hold:

(i) $\lim_{t\to\infty}(1/t)\log\det\nabla_x\varphi(\omega,t)=\sum_{i=1}^{r(\omega,x)}k_i(\omega,x)\lambda_i(\omega,x)$,

(ii) $\lim_{t\to\infty}(1/t)\log J_t(K)$ *exists for a k-dimensional subspace K of T_xD*,

(iii) *T_xD decomposes into a direct sum of invariant measurable subspaces $E_i(\omega,x)$, $i=1,\dots,r(\omega,x)$, with dimension of $E_i(\omega,x)$ equal to $k_i(\omega,x)$, and we have*

$$\lim_{t\to\infty}\frac{1}{t}\log\|\nabla_x\varphi(\omega,t)e\|=\lambda_i(\omega,x) \tag{3.8}$$

uniformly for $e\in E_i(\omega,x)$ with $\|e\|=1$.

Proof. Statements (i), (ii) and (iii) are the statements of Theorems 1, 2 and 4 in Section 3 of Oseledec (1968), respectively, which are altogether called MET. We have (Oseledec, 1968, p. 214).

$$\frac{\mathrm{d}}{\mathrm{d}t}\log\|\nabla_x\varphi(\omega,t)\|\leqslant\|\nabla u(\omega,x,t)\|$$

and hence

$$\sup_{0\leqslant t\leqslant 1}\log^+\|[\nabla_x\varphi(\omega,t)]^{\pm 1}\|\leqslant\int_0^1\|\nabla u(\omega,x,t)\|\,\mathrm{d}t,$$

where $\log^+=\max(\log,0)$. Then, stationarity of u and condition (3.7) imply that

$$\sup_{0\leqslant t\leqslant 1}\log^+\|[\nabla.\varphi(\cdot,t)]^{\pm 1}\|\in L^1(\Omega\times D,\mu),$$

which is the condition of MET. Moreover in (i), the logarithm is well defined as $\varphi(\omega,t)$ is a homeomorphism almost surely, hence its Jacobian, the determinant of the matrix $\nabla_x\varphi(\omega,t)$, is positive. □

As a corollary, for μ-almost every (ω,x), and $e\in E_i(\omega,x)$, we have

$$\sum_{i=1}^{r(\omega,x)}k_i(\omega,x)\lambda_i(\omega,x)=\lim_{t\to\infty}\frac{1}{t}\int_0^t\operatorname{div}u(\omega,\varphi(\omega,s)x,s)\,\mathrm{d}s \tag{3.9}$$

which follows from the well known identity

$$\det\nabla_x\varphi(\omega,t)=\exp\int_0^t\operatorname{div}u(\omega,\varphi(\omega,s)x,s)\,\mathrm{d}s \tag{3.10}$$

(Coddington and Levinson, 1955, Theorem 1.7.2).

If v is the *unique* stationary distribution for Z, then the process Z is ergodic and hence μ is an ergodic measure for Θ_t. Then, from ergodic theorem r, k_i, λ_i, $i=1,\ldots,r$ are free of (ω,x) and the sum of the Lyapunov exponents is given by

$$\sum_{i=1}^{r}k_i\lambda_i=\int_E v(\mathrm{d}y,\mathrm{d}x)\operatorname{div}y(x)$$

in view of (3.9). Although the Lyapunov exponents do not depend on x and are deterministic in this case, the Oseledec spaces E_i, $i=1,\ldots,r$ depend on x and are random.

The actual value of the limit in (ii) of Theorem 3.1 depends on the position of K relative to Oseledec spaces E_i, $i=1,\ldots,r$. However, this limit is as large as possible in the following sense. We demonstrate for $k=1$. Suppose that $D\subset\mathbb{R}^d$ is the closure of a ball around 0; so for each $x\in D$, $T_xD=\mathbb{R}^d$. Let σ be the area measure on S^{d-1}. Then, for μ almost every (ω,x) and for σ almost every $e\in S^{d-1}$, the limit in (3.8) is equal to $\lambda_1(\omega,x)$. This has been shown in the context of Brownian flows in Baxendale (1986), by using results from linear algebra.

4. Homogeneous and incompressible case

In this section, we consider a homogeneous and incompressible flow on $\mathbb{R}^d$ to obtain Lyapunov exponents which are nonrandom and free of the initial position. Our motivation comes from the results of Zirbel (1993, Chapter 5), where Lagrangian observations are studied for homogeneous and incompressible flows in full generality.

The conditions for homogeneity stated in Theorem 2.2 will be assumed throughout this section. A flow φ is said to be *incompressible* if the Jacobian matrix $\nabla_x \varphi(t)$ is unimodular for all $t \geqslant 0$. This is a necessary and sufficient condition for the velocity field to be divergence free because of identity (3.10). It follows that the sum of the exponents is zero by (3.9).

4.1. Lagrangian velocity field

The generalized Lagrangian velocity field $\tilde{u}$ is defined by

$$\tilde{u}(y,t) = u(y + \varphi(t)x, t), \quad y \in \mathbb{R}^d, \ t \geqslant 0.$$

It is the velocity field observed from the position of a moving particle that started at x. Ergodicity of $\tilde{u}$ plays a fundamental role in the main result of this section. To prove that Lagrangian velocity is ergodic, we show that it satisfies in fact a much stronger property. Namely, it is a Markov process and its infinitesimal generator has a spectral gap.

We first prove that the generator of the Eulerian velocity has a spectral gap. Let us topologize $C^1(\mathbb{R}^d \to \mathbb{R}^d)$ with the topology of uniform convergence on compacts. Then, the generator of u can be written for functions of the form $\mathrm{e}^{\langle y,f\rangle}$ where $f\colon \mathbb{R}^d \to \mathbb{R}^d$ has compact support, $y \in C^1(\mathbb{R}^d \to \mathbb{R}^d)$, and $\langle y, f\rangle = \int_{\mathbb{R}^d} \mathrm{d}x\, y(x)\cdot f(x)$. By (2.1), with $u_0 = y$, we have

$$\begin{aligned}\mathrm{e}^{\langle u_t,f\rangle} - \mathrm{e}^{\langle y,f\rangle} &= -c\int_0^t \mathrm{d}s\,\mathrm{e}^{\langle u_s,f\rangle}\langle u_s,f\rangle + \int_0^t\int_Q N(\mathrm{d}s,\mathrm{d}q)\mathrm{e}^{\langle u_{s-},f\rangle}(\mathrm{e}^{\langle v_q,f\rangle}-1)\\ &= -c\int_0^t \mathrm{d}s\,\mathrm{e}^{\langle u_s,f\rangle}\langle u_s,f\rangle + \int_0^t \mathrm{d}s\int_Q \kappa(\mathrm{d}q)\mathrm{e}^{\langle u_s,f\rangle}(\mathrm{e}^{\langle v_q,f\rangle}-1) + M_t,\end{aligned}$$

where

$$M_t = \int_0^t\int_Q (N(\mathrm{d}s,\mathrm{d}q) - \mathrm{d}s\,\kappa(\mathrm{d}q))\mathrm{e}^{\langle u_{s-},f\rangle}(\mathrm{e}^{\langle v_q,f\rangle}-1)$$

is a martingale since $\{u_s\colon s \geqslant 0\}$ is adapted to the filtration generated by N (Ikeda and Watanabe, 1989, Theorem II.5.1). As a result, the generator L of u can be written as

$$LF(y) = -c\mathrm{e}^{\langle y,f\rangle}\langle y,f\rangle + \int_Q \kappa(\mathrm{d}q)\mathrm{e}^{\langle y,f\rangle}(\mathrm{e}^{\langle v_q,f\rangle}-1)$$

if $F(y) = \mathrm{e}^{\langle y,f\rangle}$ with fixed f.

Let ζ denote the stationary distribution of u as before, and let $E = C^1(\mathbb{R}^d \to \mathbb{R}^d)$.

Proposition 4.1. *Let $Q = \mathbb{R}^d \times \mathbb{R} \times (0,\infty)$ and let v_q be obtained from a deterministic velocity field $v \in E$ by* (2.4). *Suppose that measure κ on Q has the form $\kappa(\mathrm{d}q) = \mathrm{d}z\,\gamma(\mathrm{d}a,\mathrm{d}b)$ if $q = (z,a,b)$, where γ is a finite measure on $\mathbb{R}\times(0,\infty)$. Then, there exists a constant $k > 0$ such that*

$$-(LF,F)_{L^2(E,\zeta)} \geqslant k\|F\|^2_{L^2(E,\zeta)} \qquad (4.1)$$

for all F in the domain of L with $\int_E \zeta(\mathrm{d}y)F(y) = 0$.

Proof. It is sufficient to show (4.1) for functions of the form $F(y) = e^{\langle y,f\rangle} - \int_E \zeta(dz)e^{\langle z,f\rangle}$. (Ethier and Kurtz, 1986, p. 402) We have $LF(y) = Le^{\langle y,f\rangle}$ since $\int_E \zeta(dz)e^{\langle z,f\rangle}$ is a constant C given by $\mathbb{E}e^{\langle u_0,f\rangle}$ where u_0 has the stationary form (2.3). That is

$$C = \mathbb{E}\exp^{\langle u_0,f\rangle} = \exp\int_{-\infty}^{0} ds \int_Q \kappa(dq)\,(e^{\langle v_q,f\rangle e^{cs}} - 1).$$

It follows that the inner product $-(LF,F)$ on (E,ζ) is

$$-(LF,F) = c\int_E \zeta(dy)e^{2\langle y,f\rangle}\langle y,f\rangle - cC\int_E \zeta(dy)e^{\langle y,f\rangle}\langle y,f\rangle$$
$$-\int_Q \kappa(dq)\,(e^{\langle v_q,f\rangle} - 1)\int_E \zeta(dy)e^{2\langle y,f\rangle} + C^2\int_Q \kappa(dq)\,(e^{\langle v_q,f\rangle} - 1).$$

To evaluate the integrals on E, we use the stationary form of u_0 and take expectations as in the evaluation of C above. However, the first two integrals on E, take a few more manipulations. First, observe that

$$\mathbb{E}e^{\langle u_0,f\rangle}\langle u_0,f\rangle = \left[\frac{d}{d\xi}\,\mathbb{E}e^{\xi\langle u_0,f\rangle}\right]_{\xi=1},$$

right-hand side of which can be computed to get an integral with respect to the measure $v(dt,dq) = dt\,\kappa(dq)$. In turn, we evaluate this using integration by parts. After simplifications, we get

$$-(LF,F) = \frac{C'}{2}\int_Q \kappa(dq)(e^{\langle v_q,f\rangle} - 1)^2,$$

where

$$C' = \mathbb{E}e^{2\langle u_0,f\rangle} = \exp\int_{-\infty}^{0} ds \int_Q \kappa(dq)\,(e^{2\langle v_q,f\rangle e^{cs}} - 1).$$

On the other hand,

$$\|F\|^2 = \int_E \zeta(dy)(e^{\langle y,f\rangle} - C)^2 = \int_E \zeta(dy)e^{2\langle y,f\rangle} - C^2 = C' - C^2.$$

After simplifications, we see that statement (4.1) holds if and only if

$$\frac{\int_Q \kappa(dq)\,(e^{2\langle v_q,f\rangle} - 1)^2}{1 - \exp - \int_{-\infty}^{0} ds \int_Q \kappa(dq)\,(e^{\langle v_q,f\rangle e^{cs}} - 1)^2} \geqslant 2k \tag{4.2}$$

for some $k > 0$. Note that the denominator in (4.2) is in fact $(1 - C^2/C')$, which is $1 - \mathbb{E}e^{2\langle u_0,f\rangle}/\mathbb{E}e^{\langle u_0,f\rangle}$. This cannot be 0 since $\mathrm{Var}(e^{\langle u_0,f\rangle})$ is strictly positive due to the form of κ. For (4.2) not to hold, there must be a sequence (f_n) such that the left-hand side of (4.2) can be made arbitrarily close to 0. This can only be done by choosing f smaller and smaller in magnitude since the numerator is always positive and the denominator is bounded. But, in this case, the denominator gets very small as well. Using L'Hopital's rule, we can show that in the limit as $\varepsilon \to 0$, with $f = \varepsilon 1_K$ for some compact $K \subset \mathbb{R}^d$, the ratio is $1/2c > 0$. Hence, there must exist a $k > 0$ that satisfies (4.2). □

Under similar homogeneity and stationarity conditions on the Eulerian velocity field, Fannjiang and Komorowski (1999) show that a spectral gap exists also for the generator of the Lagrangian velocity when it exists for the Eulerian velocity. This is basically because the generator of $\tilde{u}$ can formally be written as

$$\tilde{L}F = LF + (u, \nabla F). \tag{4.3}$$

There, the Markovian velocity is continuous in time and the model includes a diffusion term. However, the proofs of their results which are relevant to our case remain valid with a velocity field as u and no diffusion term.

Eq. (4.3) is derived in an explicit form in Zirbel (2000a) for a flow on a homogeneous and Markovian velocity field. That is, homogeneity is sufficient to conclude that the generalized Lagrangian velocity is also Markovian, whether the flow is compressible or incompressible. Incompressibility plays a role in the stationarity of the Lagrangian velocity. For our model, a derivation of (4.3) exists for certain functions F in Çağlar (1997, Chapter 5). Another special case is studied in Carmona and Xu (1997) for a class of Gaussian flows.

Proposition 4.2. *Let $Q = \mathbb{R}^d \times \mathbb{R} \times (0,\infty)$ and let v_q be obtained from a deterministic velocity field $v \in E$ by* (2.4), *with $\nabla \cdot v = 0$. Suppose that the measure κ on Q has the form $\kappa(\mathrm{d}q) = \mathrm{d}z\,\gamma(\mathrm{d}a, \mathrm{d}b)$ if $q = (z, a, b)$, where γ is a finite measure on $\mathbb{R} \times (0,\infty)$. If u_0 has the stationary distribution ζ, then the process $\{\tilde{u}_t\colon t \geqslant 0\}$ is also stationary and ergodic.*

Proof. It is shown in Zirbel (1993, Chapter 5; 2000b) that $\tilde{u}$ has the same stationary distribution ζ as u when u is homogeneous and divergence free. Since L satisfies (4.1) by Proposition 4.1, a spectral gap exists also for $\tilde{L}$ (Fannjiang and Komorowski, 1999, Remark 2, Proposition 6; Rosenblatt, 1971). It follows that $\tilde{u}$ is strong mixing (Doukhan, 1994, p. 3, 20), and hence ergodic. □

Corollary 4.3. *The process $\{\nabla u(\varphi(t)x, t)\colon t \geqslant 0\}$ is stationary and ergodic.*

Proof. Let $\tilde{\vartheta}$ denote the shift corresponding to the stationary process $\tilde{u}$. We have

$$\begin{aligned}\partial_j \tilde{u}^i(\tilde{\vartheta}_s\omega, z, t) &= \frac{\partial}{\partial z^j}\left[\tilde{u}^i(\tilde{\vartheta}_s\omega, z, t)\right] = \frac{\partial}{\partial z^j}\left[\tilde{u}^i(\omega, z, s+t)\right] \\ &= \partial_j \tilde{u}^i(\omega, z, s+t)\end{aligned}$$

for each $s, t \geqslant 0$, which means $\{\nabla\tilde{u}(z,t)\colon t \geqslant 0\}$ is stationary. In particular with $z = 0$, we get $\{\nabla u(\varphi(t)x, t)\colon t \geqslant 0\}$ is stationary with respect to the ergodic shift $\tilde{\vartheta}$. □

4.2. Lyapunov exponents

We consider the linearized flow equation

$$\frac{\mathrm{d}}{\mathrm{d}t} Y_t = A(t) Y_t, \quad Y_0 = y \in \mathbb{R}^d, \tag{4.4}$$

where $A(t) = \nabla u(\varphi(t)x, t)$. Since the process A is stationary by Corollary 4.3, the probability measure $\mathbb{P}$ is invariant under the shift transformations $\{\tilde{\theta}_t\colon t \in \mathbb{R}\}$. As a result, the dynamical system considered in this section is $(\Omega, \sigma(\tilde{u}), \mathbb{P}, (\tilde{\theta}_t)_{t\in T})$. Let M denote the fundamental matrix of system (4.4). By the uniqueness of solutions to (4.4) and the stationarity of A, we have

$$M_{t+s}(\omega) = M_t(\tilde{\theta}_s\omega) \circ M_s(\omega)$$

and hence M is a cocycle over the shift $\tilde{\theta}$. From Eq. (3.6), we see that $M = \nabla_x\varphi(t)$ and the Lyapunov exponents are defined by (3.5) as before. In this setup, a finite invariant measure μ on $\Omega \times \mathbb{R}^d$ is not required basically because of the homogeneity of u. In fact, the Lebesgue measure on $\mathbb{R}^d$ is invariant for the particle path $\{X_t\colon t \geqslant 0,\ X_0 = x\}$ because of incompressibility. We are ready to apply MET.

Theorem 4.4. *Let $Q = \mathbb{R}^d \times \mathbb{R} \times (0,\infty)$ and let v_q be obtained from a deterministic velocity field $v \in E$ by (2.4), with $\nabla \cdot v = 0$. Suppose that the measure κ on Q has the form $\kappa(\mathrm{d}q) = \mathrm{d}z\,\gamma(\mathrm{d}a, \mathrm{d}b)$ if $q = (z, a, b)$, where γ is a finite measure on $\mathbb{R} \times (0,\infty)$. Moreover, suppose that*

$$\int_{\mathbb{R}^d} \mathrm{d}z \int_{\mathbb{R}\times(0,\infty)} \gamma(\mathrm{d}a, \mathrm{d}b)\, \frac{|a|}{b^{d+1}} \|\nabla v(z)\| < \infty.$$

Then, there exists an almost sure set $\Omega_0 \subset \Omega$ such that

$$r(\omega, x), \quad \lambda_i(\omega, x), \quad k_i(\omega, x), \quad i = 1, \ldots, r(\omega, x)$$

are the same for all $\omega \in \Omega$ and all $x \in \mathbb{R}^d$, and on Ω_0 for every $x \in \mathbb{R}^d$

(i) $\sum_{i=1}^r k_i\lambda_i = 0$,

(ii) $\lim_{t\to\infty}(1/t)\log J_t(K)$ *exists for* $K \in G_k(\mathbb{R}^d)$,

(iii) $\mathbb{R}^d$ *decomposes into a direct sum of invariant measurable subspaces $E_i(\omega, x)$, $i = 1, \ldots, r$, with dimension of $E_i(\omega, x)$ being k_i, and uniformly for $e \in E_i(\omega, x)$ with $\|e\| = 1$, we have*

$$\lim_{t\to\infty} \frac{1}{t} \log \|\nabla_x\varphi(\omega, t)\, e\| = \lambda_i.$$

Proof. We have

$$\begin{aligned}
\mathbb{E}\|\nabla u(0,0)\| &\leqslant \mathbb{E} \int_{(-\infty,0]\times Q} N(\mathrm{d}s, \mathrm{d}q)\mathrm{e}^{cs} \|\nabla v_q(0)\| \\
&= \int_{-\infty}^0 \mathrm{d}s\, \mathrm{e}^{cs} \int_{\mathbb{R}^d} \mathrm{d}z \int_{\mathbb{R}\times(0,\infty)} \gamma(\mathrm{d}a, \mathrm{db}) \left\| \nabla \left[av\left(\frac{-z}{b}\right)\right]\right\| \\
&= \frac{1}{c} \int_{\mathbb{R}^d} \mathrm{d}z \int_{\mathbb{R}\times(0,\infty)} \gamma(\mathrm{d}a, \mathrm{d}b) \frac{|a|}{b} \left\| \nabla v \left(\frac{-z}{b}\right)\right\| \\
&= \frac{1}{c} \int_{\mathbb{R}^d} \mathrm{d}z \int_{\mathbb{R}\times(0,\infty)} \gamma(\mathrm{d}a, \mathrm{d}b) \frac{|a|}{b^{d+1}} \|\nabla v(z)\|
\end{aligned}$$

which is finite by the hypothesis. Then as in the proof of Theorem 3.1, we get

$$\mathbb{E} \sup_{0\leqslant t\leqslant 1} \log^+ \|[\nabla_x\varphi(t)]^{\pm 1}\| \leqslant \mathbb{E} \int_0^1 \|\nabla u(x,t)\| < \infty$$

using stationarity and homogeneity of u, as well. When the basic vortex v is divergence free, so is u_t, $t \geqslant 0$, and thus $\det \nabla_x \varphi(t) = 1$ for all $t \geqslant 0$. Then, the results follow from MET. Since the stationary process $A = \{\nabla u(\varphi(t)x, t): t \geqslant 0\}$ is ergodic by Corollary 4.3, the values λ_i, k_i, r_i are nonrandom. What is more, they do not depend on the starting point x, as the probability law of A is free of x because of homogeneity. The Oseledec spaces in general depend on x and are random, however their laws should be free of x because of homogeneity. □

4.3. Computations

Simulation of homogeneous and incompressible flows on $\mathbb{R}^2$ has been studied in Çağlar (2000) in detail. Following the same setup, we compute the top Lyapunov exponent in $\mathbb{R}^2$ for a variety of scales of motion in this part. The basic vortex v is taken to be rotation on the unit disk with a smooth profile. Namely, the speed of rotation is a differentiable function of the distance from the origin, and is zero at the origin and the boundary of the unit disk. Taking v to be rotation makes the flow isotropic in particular.

Let $J_t = \nabla_x \varphi(t)$. The top Lyapunov exponent is obtained as the limit

$$\lambda_1 = \lim_{t \to \infty} \frac{1}{t} \log \|J_t e\| \tag{4.5}$$

when the initial unit vector e is chosen independently from the velocity field. The numerical computation of limit (4.5) requires attention for the control of numerical instabilities, because the quantity $\|J_t e\|$ grows exponentially fast in time. We apply the projection procedure proposed in Talay (1991) for this purpose. Let x_{t_k} be the particle position at time t_k found by the Euler scheme

$$x_{t_k} = x_{t_{k-1}} + u(x_{t_{k-1}}, t_{k-1})(t_k - t_{k-1}),$$

where $t_k - t_{k-1}$ is small enough and no interval $[t_{k-1}, t_k]$ contains an arrival time (of a vortex). Then, the algorithm is as follows. Let $S_0 = e$ with $\|e\| = 1$, and let $k = 0$.

1. Compute

$$M_{k+1} = I + \nabla u(x_{t_k}, t_k)(t_{k+1} - t_k),$$

$$S'_{k+1} = M_{k+1} S_k,$$

$$S_{k+1} = S'_{k+1} / \|S'_{k+1}\|,$$

$$\lambda_1^{(t_n)} = \frac{1}{t_n} \sum_{k=0}^{n-1} \log \|M_{k+1} S_k\|.$$

2. Stop if $|\lambda_1^{(t_k)} - \lambda_1^{(t_{k-1})}| / \lambda_1^{(t_{k-1})}$ is less than 0.01. Otherwise, set $k = k + 1$ and go to 1.

We now show that $\sum_{k=0}^{n-1} \log \|M_{k+1} S_k\|$ is equal to $\|J_{t_n} e\|$. We compute $J_{t_{k+1}} e$ by

$$J_{t_{k+1}} e = J_{t_k} e + \nabla u(x_{t_k}, t_k) J_{t_k} e (t_{k+1} - t_k)$$

$$= M_{k+1} J_{t_k} e$$

by the Euler scheme for $\mathrm{d}J_t = \nabla u(X_t, t)J_t\,\mathrm{d}t$, $J_0 = I$. Denote $J_{t_n}e$ by V_n, then

$$\log\|V_n\| = \log\left[\frac{\|V_n\|}{\|V_{n-1}\|}\cdot\frac{\|V_{n-1}\|}{\|V_{n-2}\|}\cdots\frac{\|V_1\|}{\|V_0\|}\right]$$

$$= \sum_{k=0}^{n-1}\log\frac{\|V_{k+1}\|}{\|V_k\|} = \sum_{k=0}^{n-1}\log\frac{\|M_{k+1}V_k\|}{\|V_k\|}$$

$$= \sum_{k=0}^{n-1}\log\|M_{k+1}S_k\|,$$

where $S_k = V_k/\|V_k\|$. The only remaining part of the algorithm is, then, the update of S_k. For this, note that

$$S'_{k+1} \equiv \frac{V_{k+1}}{\|V_k\|} = M_{k+1}\frac{V_k}{\|V_k\|} = M_{k+1}S_k.$$

Then, S_{k+1} is found by normalizing S'_{k+1}, that is, $S_{k+1} = S'_{k+1}/\|S'_{k+1}\|$.

We compute λ_1 for a range of values of the parameters, the most obvious of which is the decay parameter c. We select the distributions of the amplitude a and the dilation factor b to be independent. Then, the mean measure of N can be written as

$$\lambda\,\mathrm{d}t\,\mathrm{d}z\,\alpha(\mathrm{d}a)\beta(\mathrm{d}b),$$

where α and β are probability measures and λ is the arrival rate per unit space–unit time. We fix a measure β with a bounded support so that the velocity field can be generated over a finite region completely. We choose α to be the uniform distribution on $[-\hat{a}, \hat{a}]$, hence introduce a single parameter $\hat{a}$. Our experiments show that the rate of convergence of limit (4.5) greatly varies over the range of these parameters. So, we handle each set of parameters separately and watch for convergence. The guideline in this process is the identification of the physical scales of the problem, which will also determine the range of values of the parameters covered for the computation of λ_1 below. The physical scales are the typical length scale

$$l = K_1\left(\int \beta(\mathrm{d}b)b^2\right)^{1/2},$$

the typical time scale

$$\tau_T = \frac{K_2}{\hat{a}}\sqrt{\frac{c}{\lambda}}, \tag{4.6}$$

where K_1, K_2 are constants, and the (Eulerian) decay time scale $\tau_E = 1/c$. The constants K_1, K_2 depend on the speed profile of v, which is fixed. Roughly, the particle moves l units in space in τ_T units of time because the ratio l/τ_T is the typical velocity calculated as the square root of the sum of the variances of $u^1(0,0)$ and $u^2(0,0)$ following the usual convention. The derivations can be found in Çağlar (2000).

We span a variety of motions by fixing the length scale l around 0.24, but changing the typical time τ_T and the decay time τ_E with respect to each other and the unit

Table 1
Separation of scales for typical time, decay time and unit time

$\tau_T = 0.01$	$\tau_T = 0.1$	$\tau_T = 1$	$\tau_T = 10$	$\tau_T = 100$
$\tau_T \ll \tau_E \ll 1$	$\tau_T \ll 1 \ll \tau_E$ $\tau_T \ll \tau_E \sim 1$ $\tau_T \sim \tau_E \ll 1$ $\tau_E \ll \tau_T \ll 1$	$\tau_E \ll \tau_T \sim 1$ $\tau_E \sim \tau_T \sim 1$ $1 \sim \tau_T \ll \tau_E$	$1 \sim \tau_E \ll \tau_T$ $\tau_E \ll 1 \ll \tau_T$ $1 \ll \tau_E \sim \tau_T$ $1 \ll \tau_T \ll \tau_E$	$1 \ll \tau_E \ll \tau_T$

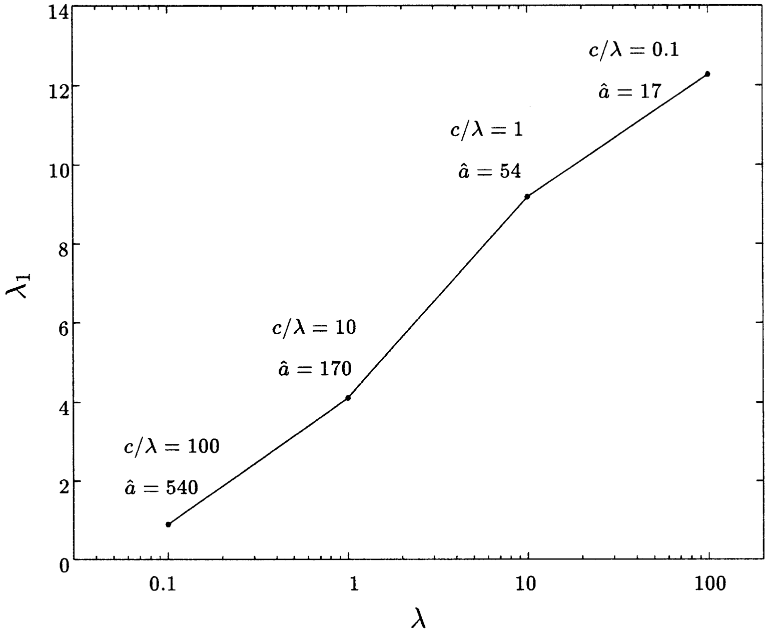


Fig. 1. The top exponent λ_1 versus arrival rate λ for $\tau_T = 0.01$. Here $c = 10$, and the values of $\hat{a}$ and c/λ are depicted on the graph for each point.

time 1. Table 1 summarizes all possible arrangements where the relation $a \ll b$ means $b = 10a$ for $a, b \in \mathbb{R}$. Each entry in Table 1 fixes the value of τ_E, hence c, and τ_T. In view of (4.6), this leaves freedom for choice of the values of λ and $\hat{a}$. Our approach is to sample for the ratios

$$\frac{c}{\lambda} = 0.1, 1, 10, 100$$

but for values of λ between 0.1 and the maximum c in each column in Table 1. This amounts to omitting some of these ratios for some values of τ_T. For $\tau_T = 0.01$, we have $c = 10$, and we sample for $\lambda = 0.1, 1, 10, 100$ where $\lambda = 100$ is an exception to this rule. The result is illustrated in Fig. 1. The values of the top Lyapunov exponent for $\tau_T = 0.1$ and the pairs (c, λ) at which it is computed are illustrated in Figs. 2a and b, respectively. Similarly, Figs. 3a, b and 4a, b are for $\tau_T = 1$ and 10, respectively. For graphical reasons Fig. 4 does not include the pair $(c, \lambda) = (0.01, 0.1)$, but it is taken into account in the final figure below. The points marked in Figs. 2b, 3b and 4b

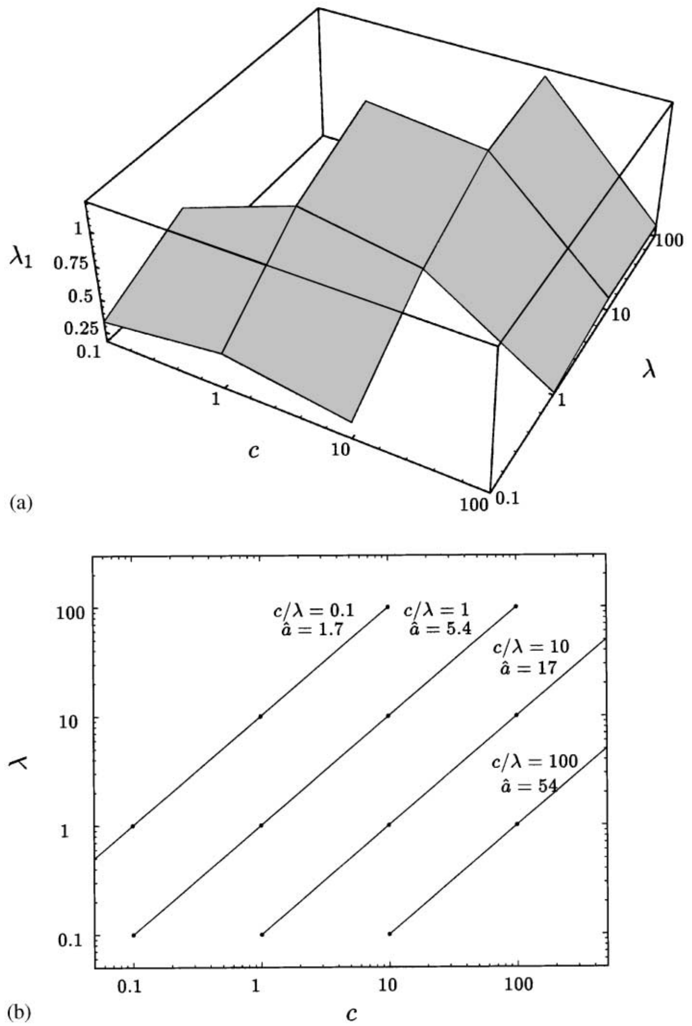
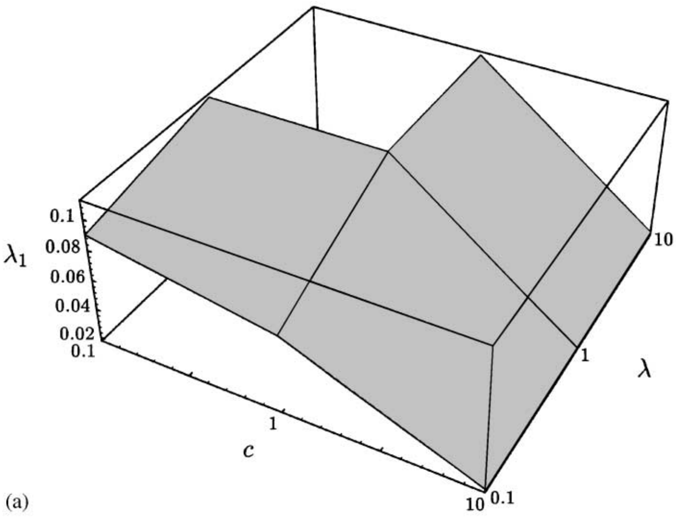


Fig. 2. (a) The top exponent λ_1 versus a range of values of (c, λ) for $\tau_T = 0.1$. (b) Marked points indicate the sampled pairs (c, λ) for $\tau_T = 0.1$.

denote the pairs (c, λ) for which λ_1 is computed, and the lines drawn on these graphs indicate the points of constant $\hat{a}$ and c/λ. Note that for each τ_T, $\hat{a}$ is the same for each value of c/λ. For $\tau_T = 100$, we have $c = 0.1$ and sample for only $\lambda = 0.1$; the result is $\lambda_1 = 2.4 \times 10^{-4}$. Note that there is no unique behavior of λ_1 with respect to c and λ throughout the different values of τ_T. The convergence occurs somewhere between time $10^3\tau_T$ to time $10^4\tau_T$ in all these computations. The stopping criterion given in Step 2 of the algorithm above, is observed for at least $10^2\tau_T$ to make sure that convergence is obtained. In general, it is not possible to compute a bound on the error of the approximation of λ_1. In the case of a dynamical system driven by a number of finite state Markov chains, Ezzine (1996) gives a stopping rule that depends on the error of the approximation.



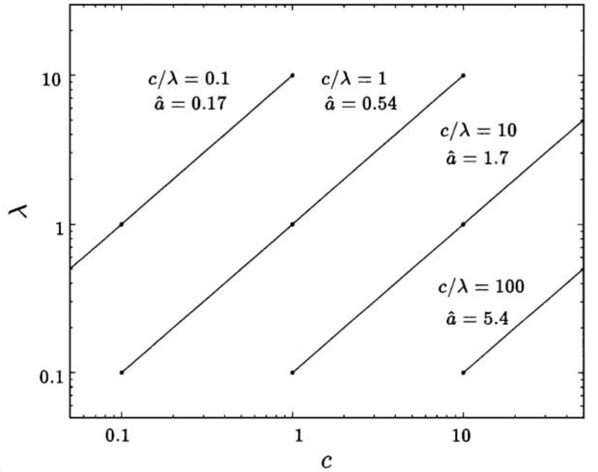


Fig. 3. (a) The top exponent λ_1 versus a range of values of (c, λ) for $\tau_T = 1$. (b) Marked points indicate the sampled pairs (c, λ) for $\tau_T = 1$.

The ultimate summary of the results is given in Fig. 5 where we plot the maximum and minimum values of the top Lyapunov exponent for each value of the typical time in a log–log scale. First, λ_1 appears to be strictly positive, hence we conjecture that the gradient vectors in homogeneous and incompressible flows grow exponentially fast. Second, our computations support that the dependence of the magnitude of λ_1 on the parameters of the model can be explained through τ_T. Although the ranges of λ_1 for different τ_T may intersect, the exponent λ_1 increases as τ_T decreases for a wide range of combinations of parameters λ, c and $\hat{a}$. Finally, our computations span a wide range of values of λ_1, from 10^{-4} to 10^1 in magnitude.

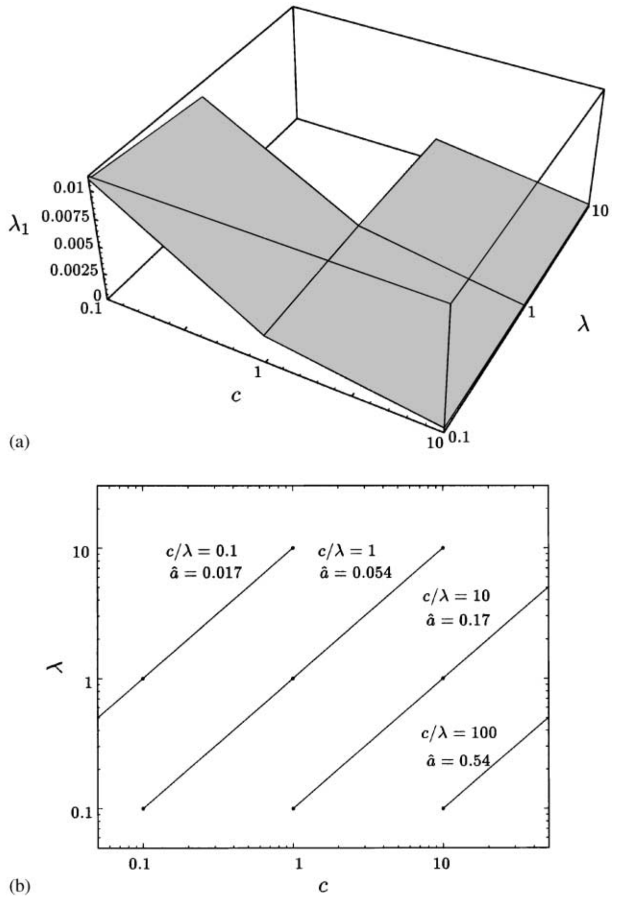


Fig. 4. (a) The top exponent λ_1 versus a range of values of (c, λ) for $\tau_T = 10$. (b) Marked points indicate the sampled pairs (c, λ) for $\tau_T = 10$.

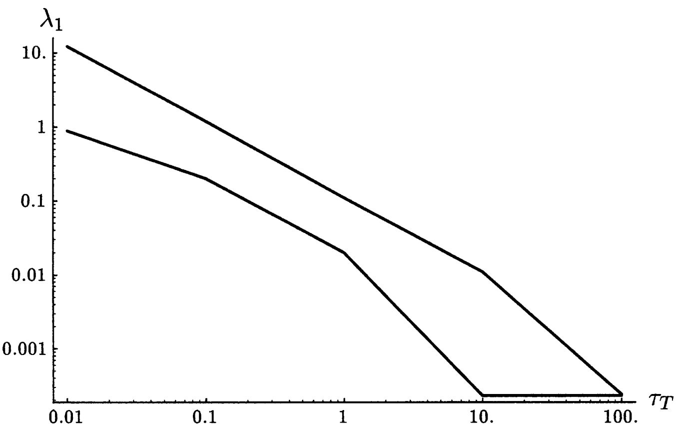


Fig. 5. The top exponent λ_1 versus the typical time τ_T. Upper and lower curves are for maximum and minimum values obtained in computations, respectively.

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